FEBRUARSKI ROK

РЕЗУЛТАТИ ИСПИТА МАТЕМАТИКА 2 ПИСАНИ ДЕО (група xyz)

Број индекса	**Презиме и име**	**К1**
20240061	Глишовић Теодора	29
20240147	Ђурдић Тамара	21
20240133	Живановић Миа	18.5
20240035	Пешић Матија	17.5
20240153	Мирић Наталија	16
20240109	Чекановић Ана	14
20240050	Ненадовић Лука	10.5
20240169	Лијескић Рења	9.5
20240200	Марија Петровић	8.5
20240053	Колић Огњен	6.5
20240150	Меза Јована	4.5
20240036	Станић Анђела	4.5
20240131	Крстић Анђела	3
20240181	Тешић Анђела	Испит није положен због пушкица у вежбанци

У прилогу се налазе задаци са решењима.
Увид у радове у четвртак у 13:30.

Интегрални колоквијум - писани део из Математике 2 на ТМФ-у у Београду **група** xyz

1. Одредити: а) $\boxed{4}$ $\int \sqrt{x^2-4x+13}\,dx$; б) $\boxed{4}$ $\int \frac{5x^2+6x}{(x-2)(x^2+4x+20)}dx$;

2. Израчунати: а) $\boxed{4}$ $\int\limits_{\frac{1}{10}}^{\frac{1}{5}} \arccos(5x)\,dx$; б) $\boxed{4}$ $\int\limits_{-\frac{\pi}{4}}^{0} \frac{dx}{\sin^2 x+2\sin x\cos x-3\cos^2 x}$.

3. а) $\boxed{4}$ Одредити област дефинисаности функције $f(x,y)=arcsin(x+y-2)+ln(x+1)-\sqrt{4y-x^2-y^2}$.
б) $\boxed{4}$ Наћи парцијалне изводе $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ и тотални диференцијал првог реда функције $z(x,y)=x^2y\ln\left(\frac{1}{y}+x^2y\right)$ у тачки $N(-1,1)$.

4. $\boxed{4}$ Одредити локалне екстремне вредности функције $f(x,y)=\frac{1}{x-1}+\frac{x-1}{y}+y$.

1. a) $\int\sqrt{x^2-4x+13}\,dx = \int\sqrt{x^2-4x+13}\cdot\frac{\sqrt{x^2-4x+13}}{\sqrt{x^2-4x+13}}\,dx =$

одакле?

$$\Rightarrow \int\frac{x^2-4x+13}{\sqrt{x^2-4x+3}}\,dx = (ax+b)\cdot\sqrt{x^2-4x+13}+\lambda\int\frac{dx}{\sqrt{x^2-4x+13}} \quad /'$$

$$\frac{x^2-4x+13}{\sqrt{x^2-4x+3}} = (ax+b)'\sqrt{x^2-4x+13}+(ax+b)\left(\sqrt{x^2-4x+13}\right)'+\lambda\cdot\frac{1}{\sqrt{x^2-4x+13}}$$

$$\frac{x^2-4x+13}{\cancel{\sqrt{x^2-4x+13}}} = a\sqrt{x^2-4x+13}+(ax+b)\frac{1}{\cancel{2}}\cdot\frac{\overset{x-2}{\cancel{(2x-4)}}}{\cancel{\sqrt{x^2-4x+3}}}+\frac{\lambda}{\cancel{\sqrt{x^2-4x+13}}} \quad /\cdot\sqrt{}$$

$$x^2-4x+13 = a(x^2-4x+13)+(ax+b)(x-2)+\lambda$$

$$x^2-4x+13 = ax^2-4ax+13a+ax^2-2ax+bx-2b+\lambda$$

$$x^2-4x+13 = 2ax^2-6ax+bx+13a-2b+\lambda$$

$x^2:\quad 1=2a \quad\Rightarrow\quad a=\frac{1}{2} \qquad b=-1$

$x:\quad -4=-6a+b \quad\Rightarrow\quad b=6a-4=6\cdot\frac{1}{2}-4=3-4=-1$

$x^0:\quad 13=13a-2b+\lambda \quad\Rightarrow\quad \lambda=13-13a+2b$

$= 13-13\cdot\frac{1}{2}-2 = \frac{13}{2}-\frac{4}{2}=\frac{9}{2}$

$\lambda=\frac{9}{2}$

$$\Rightarrow \int\sqrt{x^2-4x+13}\,dx = \left(\frac{1}{2}x-1\right)\cdot\sqrt{x^2-4x+13}+\frac{9}{2}\underbrace{\int\frac{dx}{\sqrt{x^2-4x+13}}}$$

$$\int\frac{dx}{\sqrt{x^2-4x+13}} = \int\frac{dx}{\sqrt{(x-2)^2+3^2}} = \ln\left|x-2+\sqrt{x^2-4x+13}\right|$$

$$\Rightarrow \int \sqrt{x^2-4x+13}\,dx = \left(\frac{1}{2}x-1\right)\sqrt{x^2-4x+13} + \frac{9}{2}\ln\left|x-2+\sqrt{x^2-4x+13}\right|$$

b) $\int \frac{5x^2+6x}{(x-2)(x^2+4x+20)}\,dx = I$

$$\underbrace{\frac{5x^2+6x}{(x-2)(x^2+4x+20)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+4x+20}}$$

$$= \frac{A(x^2+4x+20) + (x-2)(Bx+C)}{(x-2)(x^2+4x+20)}$$

$$\Rightarrow 5x^2+6x = Ax^2+4Ax+20A+Bx^2+Cx-2Bx-2C$$

x^2: $5x^2 = Ax^2+Bx^2 \Rightarrow 5 = A+B \Rightarrow B = 5-A$

x: $6x = 4Ax+Cx-2Bx \Rightarrow 6 = 4A+C-2B$

x^0: $0 = 20A - 2C$ $\quad 0 = 20A-2C$

$\Rightarrow 2C = 20A \quad /:2$

$C = 10A$

$6 = 4A+10A-2(5-A)$

$6 = 14A - 10 + 2A$

$6 = 16A - 10 \Rightarrow A = 1 \quad B = 4 \quad C = 10$

$$I = \overbrace{\int \frac{dx}{x-2}}^{I_1} + \int \frac{4x+10}{x^2+4x+20}\,dx$$

$$= \underbrace{\int \frac{dx}{x-2}}_{I_1} + \underbrace{4\int \frac{x}{x^2+4x+20}\,dx}_{I_2} + \underbrace{10\int \frac{dx}{x^2+4x+20}}_{I_3}$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\,\mathrm{arctg}\,\frac{x}{a}$$

$I_1 = \ln|x-2|$

I_3: $\int \frac{dx}{x^2+4x+20} = \int \frac{dx}{x^2+4x+4+16} = \int \frac{dx}{(x+2)^2+4^2} = \frac{1}{4}\,\mathrm{arctg}\,\frac{x+2}{4}$

$(x+2)^2 = x^2+4x+4$

I_2: $(x^2+4x+20)' = 2x+4$

$$\int \frac{x}{x^2+4x+20} = \int \frac{\frac{1}{2}(2x+4)-2}{x^2+4x+20}\,dx = \frac{1}{2}\int \frac{2x+4}{x^2+4x+20}\,dx - 2\int \frac{dx}{x^2+4x+20}$$

$$= \frac{1}{2}\ln|x^2+4x+20| - \underbrace{2\cdot\frac{1}{4}}_{-\frac{1}{2}}\operatorname{arctg}\frac{x+2}{4}$$

$$I = \ln|x-2| + 4\left[\frac{1}{2}\ln|x^2-4x+20| - \frac{1}{2}\operatorname{arctg}\frac{x+2}{4}\right] + \frac{10}{4}\operatorname{arctg}\frac{x+2}{4}$$

$$= \ln|x-2| + 2\ln|x^2-4x+20| + \frac{1}{2}\operatorname{arctg}\frac{x+2}{4} + C$$

$$= \ln|(x-2)(x^2-4x+20)^2| + \frac{1}{2}\operatorname{arctg}\frac{x+2}{4} + C$$

2. a) $$\int_{\frac{1}{10}}^{\frac{1}{5}} \arccos(5x)\,dx = \left[\begin{array}{ll} u = \arccos(5x) & dv = dx \\ du = -\frac{1}{\sqrt{1-(5x)^2}}\cdot 5\,dx & v = x \end{array}\right]$$

$$= x\cdot\arccos(5x)\Big|_{\frac{1}{10}}^{\frac{1}{5}} + 5\int_{\frac{1}{10}}^{\frac{1}{5}} \frac{x\,dx}{\sqrt{1-(5x)^2}} = x\cdot\arccos(5x)\Big|_{\frac{1}{10}}^{\frac{1}{5}} - 5\cdot\frac{1}{50}\int_{\frac{3}{4}}^{0}\frac{dt}{\sqrt{t}} =$$

$t = 1-25x^2$ | $x = \frac{1}{10} \Rightarrow t = 1-\frac{25}{100} = \frac{75}{100} = \frac{3}{4}$

$dt = -50x\,dx$ | $x = \frac{1}{5} \Rightarrow t = 1-\frac{25}{25} = 0$

$x\,dx = -\frac{1}{50}\,dt$

$$= \frac{1}{5}\underbrace{\arccos 1}_{\to 0} - \frac{1}{10}\underbrace{\arccos\frac{1}{2}}_{\pi/3} - \frac{1}{10}\cdot 2\sqrt{t}\Big|_{3/4}^{0}$$

$$= 0 - \frac{1}{10}\cdot\frac{\pi}{3} - \frac{1}{5}\left(0-\sqrt{\frac{3}{4}}\right) =$$

$$= -\frac{\pi}{30} + \frac{\sqrt{3}}{10}$$

b) $$\int_{-\frac{\pi}{4}}^{0} \frac{dx}{\sin^2 x + 2\sin x \cos x - 3\cos^2 x} = I$$

Смена: $t = \operatorname{tg} x = \frac{\sin x}{\cos x}$ $\qquad \cos^2 x = \frac{1}{1+t^2}$

$\sin^2 x = \frac{t^2}{1+t^2}$

$x = \operatorname{arctg} t$

$dx = \frac{dt}{1+t^2}$ $\qquad \cos x \sin x = \frac{t^{\cancel{2}}}{1+t^2} \cdot \frac{1}{\cancel{t}} = \frac{t}{1+t^2}$

$x = -\frac{\pi}{4} \Rightarrow t = -1, \quad x = 0 \Rightarrow t = 0$

$$\Rightarrow I = \int_{-1}^{0} \frac{1}{\frac{t^2}{\cancel{1+t^2}} + \frac{2t}{\cancel{1+t^2}} - \frac{3}{\cancel{1+t^2}}} \cdot \frac{dt}{\cancel{1+t^2}} = \int_{-1}^{0} \frac{dt}{t^2+2t-3} = \int_{-1}^{0} \frac{dt}{(t+3)(t-1)} =$$

$$\frac{1}{(t+3)(t-1)} = \frac{A}{t+3} + \frac{B}{t-1} =$$

$$= \frac{A(t-1) + B(t+3)}{(t+3)(t-1)} = \frac{At - A + Bt + 3B}{(t+3)(t-1)}$$

$$1 = At + Bt + 3B - A$$

$$\Rightarrow \begin{array}{ll} t: & 0 = A + B \\ t^0: & 1 = 3B - A \end{array} \Bigg\} \quad \begin{array}{l} B = -A \\ 1 = -3A - A = -4A \Rightarrow A = -\frac{1}{4} \quad B = \frac{1}{4} \end{array}$$

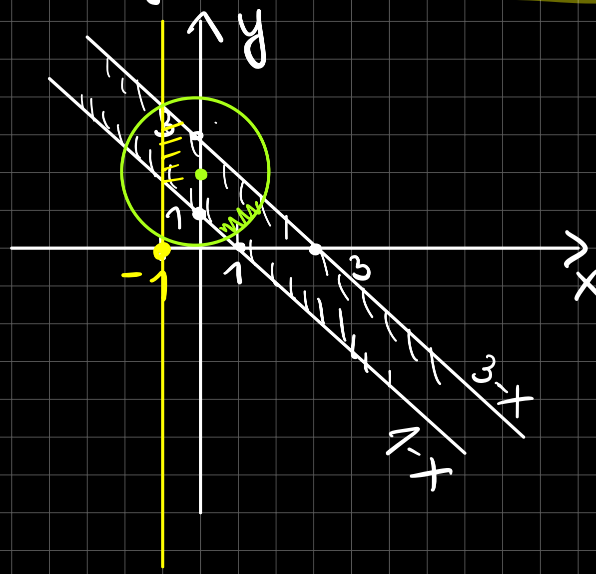
$$I = -\frac{1}{4}\int_{-1}^{0} \frac{dt}{t+3} + \frac{1}{4}\int_{-1}^{0} \frac{dt}{t-1} = \frac{1}{4} \ln\left|\frac{t-1}{t+3}\right| \Bigg|_{-1}^{0} =$$

$$= \frac{1}{4}\left[\ln\left|\frac{0-1}{0+3}\right| - \ln\left|\frac{-1-1}{-1+3}\right|\right] = \frac{1}{4}\left(\ln\left|-\frac{1}{3}\right| - \ln|-1|\right) =$$

$$= \frac{1}{4}\left(\ln\frac{1}{3} - \ln 1\right) = \frac{1}{4} \ln\frac{1}{3}$$

3. а) $f(x,y) = \arcsin(x+y-2) + \ln(x+1) - \sqrt{4y-x^2-y^2}$

I: $-1 \le x+y-2 \le 1$



$x+y-2 \le 1$

$y \le 3-x$

$x+y-2 \ge -1$

$y \ge -1+2-x$

$y \ge 1-x$

II: $x+1>0$

$x>-1$

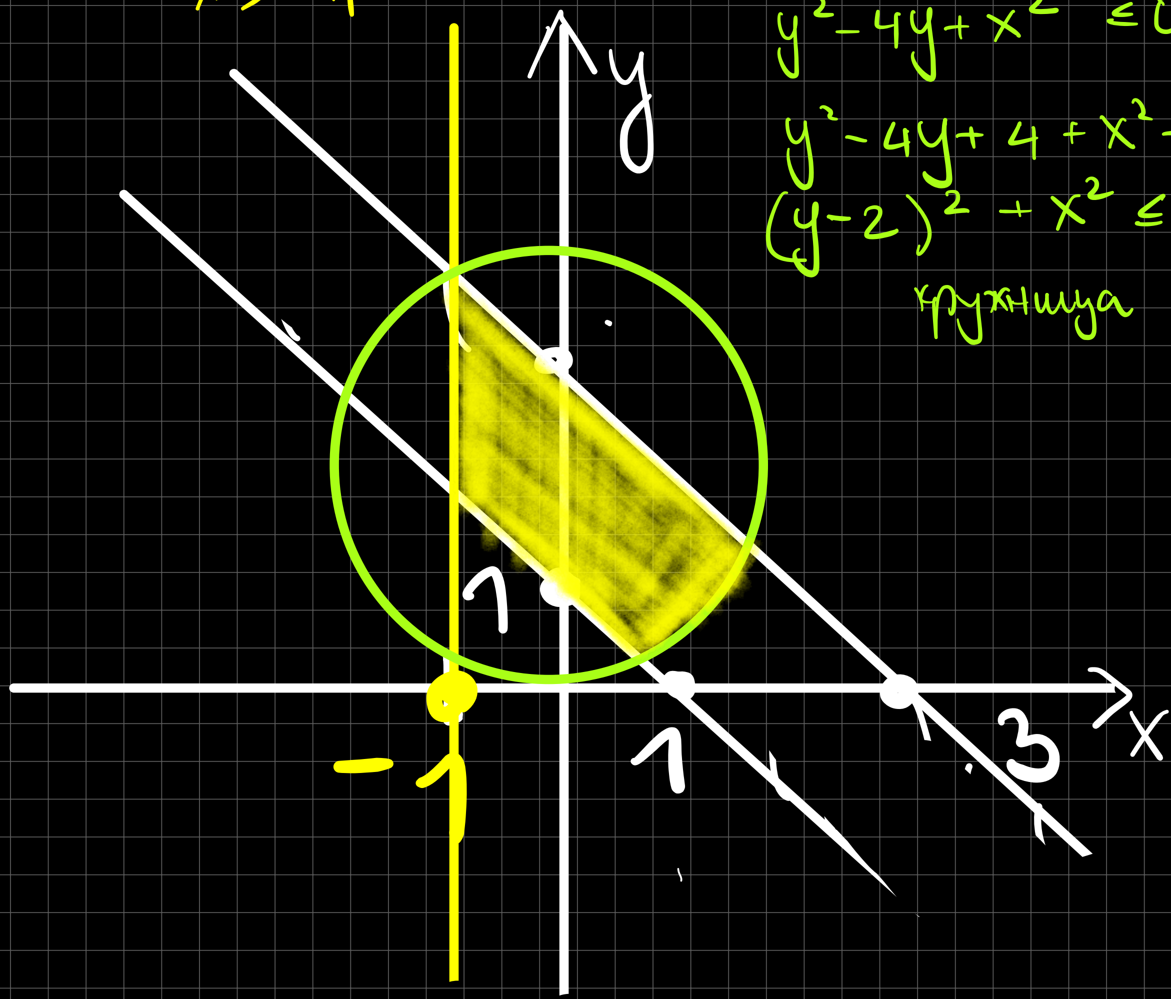
III: $4y-x^2-y^2 \ge 0$

$y^2-4y+x^2 \le 0$

$y^2-4y+4+x^2-4 \le 0$

$(y-2)^2+x^2 \le 4$

кружниця $C(0;2)$, $r=2$



$D_f = \{ (x,y) \mid y \ge 1-x \wedge y \le 3-x \wedge x>-1 \wedge x^2+(y-2)^2 \le 4 \}$

б) $\frac{\partial z}{\partial x} = ?$, $\frac{\partial z}{\partial y} = ?$, тотални диф?

$z(x,y) = x^2 y \ln\left(\frac{1}{y} + x^2 y\right)$ у $N(-1,1)$

$$\frac{\partial z}{\partial x} = 2x \cdot y \ln\left(\frac{1}{y}+x^2y\right) + x^2 y \cdot \frac{1}{\frac{1}{y}+x^2\cdot y} \cdot 2x\cdot y =$$

$$= 2xy \cdot \ln\left(\frac{1}{y}+x^2y\right) + \frac{2x^3y^2}{\frac{1+x^2y^2}{y}} =$$

$$= 2xy \ln\left(\frac{1}{y}+x^2y\right) + \frac{2x^3y^3}{1+x^2y^2}$$

$-\frac{2}{2}$

у тачки $N(-1,1)$: $2(-1)\cdot\ln(1+1) + \frac{-2}{1+1} = -2\ln 2 - 1$

$$\frac{\partial z}{\partial y} = x^2 \cdot \ln\left(\frac{1}{y}+x^2y\right) + x^2y \cdot \frac{1}{\frac{1}{y}+x^2y} \cdot \left(-\frac{1}{y^2}+x^2\right)$$

у тачки $N(-1,1)$: $1\cdot\ln(1+1) + 1\cdot\underbrace{\frac{1}{1+1}}_{\frac{1}{2}}\cdot\underbrace{(-1+1)}_{0} = \ln 2 + 0$

$$\Rightarrow \quad dz = \frac{\partial z}{\partial x}\cdot dx + \frac{\partial z}{\partial y}\cdot dy = (-2\ln 2 - 1)\,dx + \ln 2 \cdot dy$$

4. Лок. екстремуми за $f(x,y) = \frac{1}{x-1} + \frac{x-1}{y} + y$

$$\frac{\partial f}{\partial x} = -\frac{1}{(x-1)^2} + \left(\frac{x}{y}-1\right)'_x + 0 = -\frac{1}{(x-1)^2} + \frac{1}{y}$$

D_f: $x \neq 1$, $y \neq 0$

$$\frac{\partial f}{\partial y} = -\frac{x-1}{y^2} + 1$$

$$-\frac{1}{(x-1)^2} + \frac{1}{y} = 0 \quad \wedge \quad -\frac{x-1}{y^2} + 1 = 0$$

$$-y + (x-1)^2 = 0 \qquad\qquad -(x-1) + y^2 = 0 \quad \Rightarrow \quad y^2 = x-1$$

$$-y + y^4 = 0 \qquad\qquad x - 1 = 1$$

$$y(y^3-1) = 0 \qquad\qquad x = 2$$

$\Leftrightarrow$ $\cancel{y=0}$ $\vee$ $y = 1$

не припада домену

$T(2,1)$

Одређујемо Хесијан матрицу: у тачки

$$\left(\frac{\partial f}{\partial x}\right)'_x = \left(-\frac{1}{(x-1)^2}+\frac{1}{y}\right)'_x = 2\frac{1}{(x-1)^3} \overset{(2,1)}{=} 2\,\frac{1}{1} = 2>0$$

$$\left(\frac{\partial f}{\partial x}\right)'_y = \left(-\frac{1}{(x-1)^2}+\frac{1}{y}\right)'_y = -\frac{1}{y^2} \overset{(2,1)}{=} -1$$

$$\left(\frac{\partial f}{\partial y}\right)'_y = \left(-\frac{x-1}{y^2}+1\right)'_y = 2\frac{x-1}{y^3} \overset{(2,1)}{=} 2\cdot\frac{1}{1} = 2$$

$$D = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4-(-1)(-1) = 3>0$$

Како је $f''_{xx}>0$ и $D>0$ то значи да је у $f(2,1)$ локални минимум.