

# REDOVNI ROK, 2 DEO ISPITA, 21.02.2026.

Drugi deo ispita iz Matematičke obrade eksperimentalnih podataka na TMF-u u Beogradu, Redovni rok A grupa

- Četvorostrana pravilna "kocka" sa oznakama 0, 1, 2 i 3 se baca jedanput. Rezultat koji se dobije određuje koliko ćemo puta bacati novčić. Neka je  $X$  slučajna veličina koja označava ishod bacanja kockice, a  $Y$  slučajna veličina koja označava koliko puta je dobijeno pismo prilikom bacanja novčića. Odrediti:
  - 1) jednodimenzionu raspodelu slučajne promenljive  $X$
  - 5) dvodimenzionu raspodelu slučajne promenljive  $(X, Y)$  i upisati u tabelu sve  $p_{X,Y}(x, y)$ .
  - 2) jednodimenzionu raspodelu slučajne promenljive  $Y$
  - 2)  $E(Y)$ .
  - 3)  $P(X = 2|Y = 1)$  i  $P(X \geq 2|Y \geq 2)$
  - 3)  $E(X|Y = 1)$ .
  - 3)  $E(Y|X = 2)$ .
- Ako slučajna promenljiva  $X$  ima normalnu raspodelu  $X \sim N(5, 100)$ , odrediti:
  - 3)  $P(X \leq 3)$ ?
  - 4)  $P(|X| \leq 2)$ ?
  - 4)  $P(X^2 - 5X \leq 6)$ ?
  - 3) Odrediti  $x$  takvo da je  $P(X \leq x) = 0.8023$ .
  - 4) Odrediti  $x$  takvo da je  $P(X \leq x) = 0.1446$ .
- Dva igrača, A i B, igraju duel u slobodnim bacanjima. U svakoj rundi oba igrača šutiraju po jedno slobodno bacanje (nezavisno jedno od drugog). A pogadja svoje bacanje sa verovatnoćom  $p$ , B pogadja svoje bacanje sa verovatnoćom  $q$ . Poeni se dodeljuju po sledećem pravilu: ako jedan igrač pogodi, a drugi promaši u toj rundi, tada igrač koji je pogodio dobija 1 poen u toj rundi. U ostalim slučajevima niko ne osvaja poen (rezultat se ne menja). Igra se završava kada jedan igrač stekne prednost od 2 poena. Neka stanje bude razlika rezultata nakon svake runde:  $X_n = (\text{poeni}A) - (\text{poeni}B)$ 
  - 2) Koja stanja će biti završna?
  - 5) Opisati Markovljevim procesom ovu igru (odrediti sva stanja, prelaskе, kao i njihove verovatnoće).
  - 6) Odrediti verovatnoću da A pobjedi najkasnije do kraja četvrte runde.

## REŠENJA STUDENATA!

1) X - ISHOD BACANJA KOCKE  
Y - koliko puta je dobijeno pismo

a)  $X: \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$

b)  $Y: \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{8} & \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{16} & \frac{1}{16} & \frac{1}{16} & \frac{1}{16} \\ \frac{1}{32} & \frac{1}{32} & \frac{1}{32} & \frac{1}{32} \end{pmatrix}$

$P\{X=0, Y=0\} = P\{Y=0|X=0\} \cdot P\{X=0\} = 1 \cdot \frac{1}{4} = \frac{1}{4}$

$P\{X=0, Y=1\} = P\{Y=1|X=0\} \cdot P\{X=0\} = 0 \cdot \frac{1}{4} = 0$

$P\{X=0, Y=2\} = P\{Y=2|X=0\} \cdot P\{X=0\} = 0 \cdot \frac{1}{4} = 0$

$P\{X=0, Y=3\} = P\{Y=3|X=0\} \cdot P\{X=0\} = 0 \cdot \frac{1}{4} = 0$

$P\{X=1, Y=0\} = P\{Y=0|X=1\} \cdot P\{X=1\} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

$P\{X=1, Y=1\} = P\{Y=1|X=1\} \cdot P\{X=1\} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

$P\{X=1, Y=2\} = P\{Y=2|X=1\} \cdot P\{X=1\} = 0 \cdot \frac{1}{4} = 0$

$P\{X=1, Y=3\} = P\{Y=3|X=1\} \cdot P\{X=1\} = 0 \cdot \frac{1}{4} = 0$

$P\{X=2, Y=0\} = P\{Y=0|X=2\} \cdot P\{X=2\} = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

$P\{X=2, Y=1\} = P\{Y=1|X=2\} \cdot P\{X=2\} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

$P\{X=2, Y=2\} = P\{Y=2|X=2\} \cdot P\{X=2\} = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

$P\{X=2, Y=3\} = P\{Y=3|X=2\} \cdot P\{X=2\} = 0 \cdot \frac{1}{4} = 0$

$P\{X=3, Y=0\} = P\{Y=0|X=3\} \cdot P\{X=3\} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

$P\{X=3, Y=1\} = P\{Y=1|X=3\} \cdot P\{X=3\} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

$P\{X=3, Y=2\} = P\{Y=2|X=3\} \cdot P\{X=3\} = 0 \cdot \frac{1}{4} = 0$

$P\{X=3, Y=3\} = P\{Y=3|X=3\} \cdot P\{X=3\} = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$

X \ Y	0	1	2	3	Σ
0	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{15}{32}$
1	0	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{11}{32}$
2	0	0	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{5}{32}$
3	0	0	0	$\frac{1}{32}$	$\frac{1}{32}$
Σ	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	1

c)  $Y: \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{15}{32} & \frac{11}{32} & \frac{5}{32} & \frac{1}{32} \end{pmatrix}$

d)  $E(Y) = 0 \cdot \frac{15}{32} + 1 \cdot \frac{11}{32} + 2 \cdot \frac{5}{32} + 3 \cdot \frac{1}{32} = \frac{11+10+3}{32} = \frac{24}{32}$

e)  $P(X=2|Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{\frac{1}{16}}{\frac{11}{32}} = \frac{2}{11}$

f)  $P(X \geq 2|Y \geq 2) = \frac{P(X \geq 2, Y \geq 2)}{P(Y \geq 2)} = \frac{\frac{1}{32} + \frac{1}{32}}{\frac{5}{32} + \frac{1}{32}} = \frac{2}{6} = \frac{1}{3}$

$$f) E(X|Y=1) ?$$

$$X|Y=1 :$$

$$\begin{pmatrix} 1 & 2 & 3 \\ \frac{4}{11} & \frac{4}{11} & \frac{3}{11} \\ \frac{2}{11} & \frac{2}{11} & \frac{2}{11} \\ \frac{1}{11} & \frac{1}{11} & \frac{1}{11} \end{pmatrix}$$

$$\frac{1}{8} \cdot \frac{11}{32} = \frac{1}{8} \cdot \frac{11}{32} = \frac{11}{256}$$

$$\frac{2}{16} \cdot \frac{11}{11} = \frac{2}{16} \cdot \frac{11}{11} = \frac{2}{16}$$

$$E(X|Y=1) = 1 \cdot \frac{4}{11} + 2 \cdot \frac{4}{11} + 3 \cdot \frac{3}{11} = \frac{21}{11} \checkmark$$

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$$g) E(Y|X=2)$$

$$Y|X=2 : \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$Y|X=2 : \begin{pmatrix} 0 & 1 & 2 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

$$\frac{1}{16} \cdot \frac{1}{4} = \frac{1}{64}$$

$$\frac{2}{16} \cdot \frac{1}{4} = \frac{2}{64} = \frac{1}{32}$$

$$\frac{1}{16} \cdot \frac{1}{4} = \frac{1}{64}$$

$$E(Y|X=2) = \frac{1}{4} + \frac{2}{4} = 1$$

$$E(Y|X=2) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{4} = \frac{6}{4} = \frac{3}{2}$$

$$2) X \sim N(5, 100)$$

$$\mu=5, \sigma=10 \rightarrow X^* \sim N(0, 1)$$

$$a) P(X \leq 3) = P\left(\frac{X-5}{10} \leq \frac{3-5}{10}\right) = P(X^* \leq -\frac{2}{10}) = P(X^* \leq -0,2) = F(-0,2) = 1 - F(0,2) = 1 - 0,5793 = 0,4207 \checkmark$$

$$b) P(|X| \leq 2) = P(-2 \leq X \leq 2) = P\left(-\frac{2-5}{10} \leq \frac{X-5}{10} \leq \frac{2-5}{10}\right) = P\left(-\frac{7}{10} \leq X^* \leq -\frac{3}{10}\right) = P(-0,7 \leq X^* \leq -0,3) = F(-0,3) - F(-0,7) = 1 - F(0,3) - (1 - F(0,7)) = 1 - F(0,3) - 1 + F(0,7) = F(0,7) - F(0,3) = 0,7580 - 0,6179 = 0,1401$$

$$c) P(x^2 - 5x \leq 6) = P(x^2 - 5x - 6 \leq 0)$$

$$\odot (x-2)(x+3) = x^2 - 2x - 3x + 6 = x^2 - 5x + 6$$

$$\text{W} (x-6)(x+1) = x^2 - 6x + x - 6 =$$

$$= x^2 - 5x - 6$$

$$(x-6)(x+1) = 0$$



$$x_1 = 6$$

$$x_2 = -1$$

$$x^2 - 5x - 6 \leq 0$$

$$P(-1 \leq x \leq 6) = P\left(\frac{-1-5}{10} \leq \frac{x-5}{10} \leq \frac{6-5}{10}\right) = P\left(-\frac{6}{10} \leq x^* \leq \frac{1}{10}\right) =$$

$$= P(-0,6 \leq x^* \leq 0,1) = F(0,1) - F(-0,6) = F(0,1) - 1 + F(0,6)$$

$$= 0,5398 - 1 + 0,7257 = 0,2655 \checkmark$$

$$d) x = ? \quad P(x \leq x) = 0,8023$$

$$P\left(\frac{x-5}{10} \leq \frac{x-5}{10}\right) = P\left(x^* \leq \frac{x-5}{10}\right) = F\left(\frac{x-5}{10}\right) = 0,8023$$

$$\frac{x-5}{10} = 0,85$$

$$x-5 = 8,5$$

$$x = 13,5$$

$$e) x = ? \quad P(x \leq x) = 0,1446$$

$$P\left(x^* \leq \frac{x-5}{10}\right) = 0,1446$$

$$F\left(\frac{x-5}{10}\right) = 0,1446$$

$$1 - F\left(\frac{x-5}{10}\right) = 1 - 0,1446$$

$$F\left(\frac{5-x}{10}\right) = 0,8554$$

$$\frac{5-x}{10} = 1,06$$

$$5-x = 10,6$$

$$x = 5 - 10,6 = -5,6$$

- 3) a) ZAVRŠNA STANJA SU -2 - STANE U KOM IGRAČ B IMA PREDNOST OD 2 POENI  
 2 - STANE U KOM IGRAČ A IMA PREDNOST OD 2 POENI

$P(A \text{ pogodio}) = p$   $P(A \text{ promašio}) = 1-p$

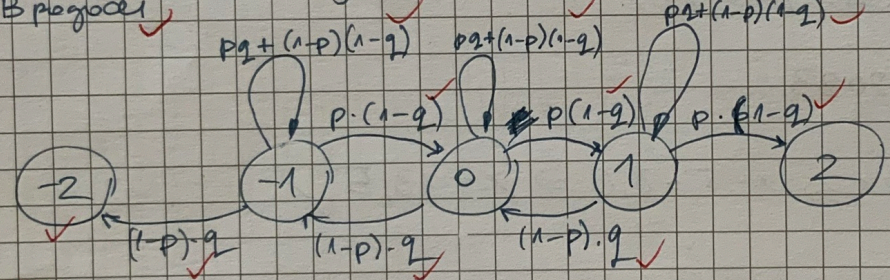
$P(B \text{ pogodio}) = q$   $P(B \text{ promašio}) = 1-q$

A pogodi  $\Rightarrow$  A dobija poen, sist ide iz st. npr. 0 u 1  $\Rightarrow P = p \cdot (1-q)$   
 B promašio

A promašio  $\Rightarrow$  ostaje isto,  $P = (1-p)(1-q)$   
 B promašio

A pogodi B pogodi  $\Rightarrow$   $P = p \cdot q$

A promašio B pogodi  $\Rightarrow$  B dobija poen, sist iz 0 u -1  $P = (1-p) \cdot q$



c) Verovatnoća DA SE STIGNE IZ 0 U 2 U NAJVIŠE 4 KORAKA

$$\begin{aligned}
 P &= P(0-0-0-1-2) + P(0-1-1-1-2) + P(0-1-2) + \\
 &+ P(0-0-1-2) + P(0-1-1-2) + P(0-1-0-1-2) + \\
 &+ P(0-0-1-1-2) + P(0-(-1)-0-1-2) = \\
 &= (pq + (1-p)(1-q))^2 \cdot (p(1-q))^2 + (p(1-q)) \cdot ((pq + (1-p)(1-q))^2 + (p(1-q))^2) + \\
 &+ (pq + (1-p)(1-q)) \cdot (p(1-q))^2 + (p(1-q)) \cdot (pq + (1-p)(1-q)) + \\
 &+ (p(1-q))^3 \cdot (1-p) \cdot q + (pq + (1-p)(1-q))^2 \cdot (p(1-q))^2 + (1-p) \cdot q \cdot p(1-q) \cdot (p(1-q))^2 \\
 &3(pq + (1-p)(1-q))^2 \cdot (p(1-q))^2 + (p(1-q))^2 (1 + 2pq(1-p)(1-q)) + \\
 &+ (p(1-q))^3 (2(1-p)q) \\
 &\hspace{15em} (2 \cdot (1-p) \cdot q)
 \end{aligned}$$

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