6. $y'' + 9y = f(t)$, $\quad f(t) = \begin{cases} 3, & 0 \le t < 1 \\ 0, & 1 \le t \end{cases}$

Нису дати почетни услови па узимамо

$y(0) = A$, $\quad y'(0) = B$

$\mathcal{L}[y''] + 9\mathcal{L}[y] = \mathcal{L}[f(t)]$

$\mathcal{L}[y''] = s^2 Y(s) - s y(0) - y'(0) = s^2 Y(s) - A \cdot s - B$

$\mathcal{L}[y] = Y(s)$

$$f(t) = 3 \begin{cases} 1, & 0 \le t < 1 \\ 0, & 1 \le t \end{cases}$$

$$\mathcal{L}[f(t)] = 3\int_0^1 e^{-st}\, dt + \int_1^{+\infty} 0 \cdot e^{-st}\, dt = -\frac{3}{s} e^{-st}\Big|_0^1 =$$

$$= -\frac{3}{s}\left(e^{-s} - 1\right) = \frac{3}{s} - \frac{3}{s} e^{-s}$$

$$s^2Y(s) - As - B + 9Y(s) = \frac{3}{s} - \frac{3}{s}\cdot e^{-s}$$

$$Y(s)(s^2+9) = \frac{3}{s} - \frac{3}{s}e^{-s} + As+B$$

$$Y(s) = \frac{3}{s(s^2+9)} - \frac{3\cdot e^{-s}}{s(s^2+9)} + \frac{As+B}{s^2+9}$$

$$y(t) = \underbrace{\mathcal{L}^{-1}\left[\frac{3}{s(s^2+9)}\right]}_{\mathrm{I}} - \underbrace{\mathcal{L}^{-1}\left[\frac{3e^{-s}}{s(s^2+9)}\right]}_{\mathrm{II}} + \underbrace{\mathcal{L}^{-1}\left[\frac{As+B}{s^2+9}\right]}_{\mathrm{III}}$$

III $\mathcal{L}^{-1}\left[\frac{As+B}{s^2+9}\right] = A\underbrace{\mathcal{L}^{-1}\left[\frac{s}{s^2+3^2}\right]}_{\cos 3t} + B\underbrace{\mathcal{L}^{-1}\left[\frac{1}{s^2+3^2}\right]}_{\frac{1}{3}\sin 3t} = A\cos 3t + \frac{B}{3}\sin 3t$

I: $\frac{1}{s(s^2+9)} = \frac{1}{9}\cdot\frac{1}{s} - \frac{1}{9}\cdot\frac{s}{s^2+9}$

$\frac{1}{9}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{1}{9}\mathcal{L}^{-1}\left[\frac{s}{s^2+3^2}\right] = \frac{1}{9} - \frac{1}{9}\cos 3t$, $\mathrm{I} = 3\left(\frac{1}{9} - \frac{1}{9}\cos 3t\right)$

II: $\mathcal{L}^{-1}\left[\frac{3e^{-s}}{s(s^2+9)}\right] \underset{\uparrow}{=} 3 f(t-1)\,u(t-1)$, $F(s) = \frac{1}{s(s^2+9)}$

8 ακρίβεια, $b=1$

$$f(t) = \frac{1}{9} - \frac{1}{9}\cos 3t \Rightarrow f(t-1) = \frac{1}{9} - \frac{1}{9}\cos 3(t-1)$$

$$\mathrm{II} = 3 f(t-1)\,u(t-1) = \left(\frac{1}{3} - \frac{1}{3}\cos 3(t-1)\right)u(t-1)$$

$$\Rightarrow y(t) = \frac{1}{3} - \frac{1}{3}\cos 3t - \left(\frac{1}{3} - \frac{1}{3}\cos 3(t-1)\right)u(t-1) + A\cos 3t + \frac{B}{3}\sin 3t$$

7. $y'(t) - y(t) + \int_0^t e^{t-x}y(x)\,dx = \cos t$, $y(0) = -1$

$g(t-x)$ $f(x)$

$\underline{\mathcal{L}[y'(t)]} - \underline{\mathcal{L}[y(t)]} + \mathcal{L}\left[\int_0^t e^{t-x}y(x)\,dx\right] = \mathcal{L}[\cos t]$

- $\mathcal{L}[y(t)] = Y(s)$
- $\mathcal{L}[y'(t)] = sY(s) - y(0) = s\cdot Y(s) + 1$
- $\mathcal{L}\left[\int_0^t \underbrace{e^{t-x}}_{g(t-x)}\cdot\underbrace{y(x)}_{f(x)}\,dx\right] = \mathcal{L}[f*g] = \underline{F(s)\cdot G(s)}$
- $f(t) = y(t) \Rightarrow F(s) = \mathcal{L}[y(t)] = \underline{Y(s)}$
- $g(t-x) = e^{t-x} \Rightarrow g(t) = e^t \Rightarrow G(s) = \mathcal{L}[g(t)] = \mathcal{L}[e^t] = \underline{\frac{1}{s-1}}$
- $\mathcal{L}[\cos t] = \frac{s}{s^2+1}$

$$s\cdot \underline{Y(s)}+1-\underline{Y(s)}+\underline{Y(s)\cdot\frac{1}{s-1}}=\frac{s}{s^2+1}$$

$$Y(s)\left(s-1+\frac{1}{s-1}\right)=\frac{s}{s^2+1}-1$$

$$Y(s)\,\frac{s^2-2s+2}{s-1}=\frac{s}{s^2+1}-1 \quad /:\frac{s^2-2s+2}{s-1}$$

$$Y(s)=\frac{s(s-1)}{(s^2+1)(s^2-2s+2)}-\frac{s-1}{s^2-2s+2}$$

$$y(t)=\underbrace{\mathcal{L}^{-1}\left[\frac{s(s-1)}{(s^2+1)(s^2-2s+2)}\right]}_{\text{I}}-\underbrace{\mathcal{L}^{-1}\left[\frac{s-1}{s^2-2s+2}\right]}_{\text{II}}$$

$$\underline{\text{I}}:\quad \mathcal{L}^{-1}\left[\frac{s(s-1)}{(s^2+1)(s^2-2s+2)}\right]$$

$$\frac{s^2-s}{(s^2+1)(s^2-2s+2)}=\frac{As+B}{s^2+1}+\frac{Cs+D}{s^2-2s+2},$$

$$\boxed{A=-\frac{3}{5} \quad B=\frac{1}{5} \quad C=\frac{3}{5} \quad D=-\frac{2}{5}}$$

$$=\frac{1}{5}\,\frac{-3s+1}{s^2+1}+\frac{1}{5}\,\frac{3s-2}{s^2-2s+2}$$

$$\Rightarrow \frac{1}{5}\mathcal{L}^{-1}\left[\frac{-3s+1}{s^2+1}\right]+\frac{1}{5}\mathcal{L}^{-1}\left[\frac{3s-2}{s^2-2s+2}\right]$$

$$=-\frac{3}{5}\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]+\frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]+\frac{3}{5}\mathcal{L}^{-1}\left[\frac{s}{s^2-2s+2}\right]-\frac{2}{5}\mathcal{L}^{-1}\left[\frac{1}{s^2-2s+2}\right]$$

$$=-\frac{3}{5}\cos t+\frac{1}{5}\sin t+\frac{3}{5}\mathcal{L}^{-1}\left[\frac{s-1+}{(s-1)^2+1}\right]+\frac{1}{5}\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$=-\frac{3}{5}\cos t+\frac{1}{5}\sin t+\frac{3}{5}e^t\cdot\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]+\frac{1}{5}e^t\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]$$

$$=-\frac{3}{5}\cos t+\frac{1}{5}\sin t+\frac{3}{5}e^t\cdot\cos t+\frac{1}{5}e^t\sin t$$

$$\underline{\text{II}}:\quad \mathcal{L}^{-1}\left[\frac{s-1}{s^2-2s+2}\right]=\mathcal{L}^{-1}\left[\frac{s-1}{(s-1)^2+1}\right]=e^t\cos t$$

$$\Rightarrow y(t)=\text{I}-\underline{\text{II}}$$

Scanned with CamScanner

8) $\mathcal{L}$/

$$x' - x + y = \sin t \quad , \quad x(0) = 1 \,, \quad y(0) = -1$$
$$y' - 2x + y = 0$$

$$\begin{cases} \mathcal{L}[x'] - \mathcal{L}[x] + \mathcal{L}[y] = \mathcal{L}[\sin t] \\ \mathcal{L}[y'] - 2\mathcal{L}[x] + \mathcal{L}[y] = 0 \end{cases}$$

$$\begin{cases} \mathcal{L}[x] = X(s) \\ \mathcal{L}[y] = Y(s) \\ \mathcal{L}[x'] = s\cdot X(s) - x(0) = sX(s) - 1 \quad 0{,}5 \\ \mathcal{L}[y'] = sY(s) - y(0) = sY(s) + 1 \quad 0{,}5 \end{cases}$$

$$\begin{cases} s\cdot X(s) - 1 - X(s) + Y(s) = \frac{1}{s^2+1} \\ sY(s) + 1 - 2X(s) + Y(s) = 0 \end{cases}$$

$$\begin{cases} (s-1)X(s) + Y(s) = \frac{1}{s^2+1} + 1 & /\cdot 2 \\ (s+1)Y(s) - 2X(s) = -1 & /\cdot (s-1) \end{cases}$$

$$\begin{cases} 2\cancel{(s-1)}X(s) + 2Y(s) = \frac{2}{s^2+1} + 2 & (1) \\ (s-1)(s+1)Y(s) - 2\cancel{(s-1)}X(s) = -s+1 & (2) \end{cases}$$

$(1)+(2)$: $\quad (2 + s^2 - 1)Y(s) = \frac{2}{s^2+1} + 2 - s + 1$

$$(s^2+1)Y(s) = \frac{2}{s^2+1} + 3 - s \quad /: (s^2+1)$$

$$(*) \quad Y(s) = \frac{2}{(s^2+1)^2} + \frac{3-s}{s^2+1} \quad 1 \,,$$

$$-2X(s) = -1 - (s+1)Y(s)$$

$$2X(s) = 1 + (s+1)Y(s)$$

$$2X(s) = 1 + \frac{2(s+1)}{(s^2+1)^2} + \frac{(s+1)(3-s)}{s^2+1}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \mathcal{L}^{-1}\left[\frac{2}{(s^2+1)^2}\right] + \mathcal{L}^{-1}\left[\frac{3-s}{s^2+1}\right] =$$
$$= 2\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right] + 3\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]$$
$$= 2\left(\frac{1}{2}\sin t - \frac{1}{2}t\cos t\right) + 3\sin t - \cos t \quad 2$$
$$= \sin t - t\cos t + 3\sin t - \cos t =$$
$$= 4\sin t - (t+1)\cos t$$

Scanned with CamScanner

$$x(t)=\mathcal{L}^{-1}[X(s)]=\mathcal{L}^{-1}\left[\frac{1}{2}\right]+\mathcal{L}^{-1}\left[\frac{s+1}{(s^2+1)^2}\right]+\frac{1}{2}\mathcal{L}^{-1}\left[\frac{(s+1)(3-s)}{(s^2+1)}\right]$$

$$\frac{(s+1)(3-s)}{(s^2+1)}=\frac{3s-s^2+3-s}{(s^2+1)}=\frac{-s^2+2s+3}{(s^2+1)}=\frac{\overbrace{-(s^2+1)}^{-s^2-1}+2s+4}{(s^2+1)}=$$

$$=-\frac{s^2+1}{(s^2+1)}+\frac{2(s+2)}{(s^2+1)}$$

$$=-1+2\frac{s+2}{s^2+1}$$

$$\Rightarrow x(t)=\cancel{\mathcal{L}^{-1}\left[\frac{1}{2}\right]}+\mathcal{L}^{-1}\left[\frac{s+1}{(s^2+1)^2}\right]\cancel{-\frac{1}{2}\mathcal{L}^{-1}[1]}+\mathcal{L}^{-1}\left[\frac{s+2}{s^2+1}\right]$$

$$=\underbrace{\mathcal{L}^{-1}\left[\frac{s+1}{(s^2+1)^2}\right]}_{I}+\underbrace{\mathcal{L}^{-1}\left[\frac{s+2}{s^2+1}\right]}_{II}$$ 1

I: $$\mathcal{L}^{-1}\left[\frac{s+1}{(s^2+1)^2}\right]=\underbrace{\mathcal{L}^{-1}\left[\frac{s}{(s^2+1)^2}\right]}_{\frac{1}{2}t\sin t}+\underbrace{\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right]}_{\text{izračunali: } \frac{1}{2}\sin t-\frac{1}{2}t\cdot\cos t}=\frac{1}{2}t\sin t+\frac{1}{2}\sin t-\frac{1}{2}t\cdot\cos t$$

II: $$\mathcal{L}^{-1}\left[\frac{s+2}{s^2+1}\right]=\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]+2\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right]=\cos t+2\sin t$$

$$\Rightarrow x(t)=\frac{1}{2}t\sin t+\frac{1}{2}\sin t-\frac{1}{2}t\cos t+\cos t+2\sin t=\frac{1}{2}t\sin t+\frac{5}{2}\sin t-\frac{1}{2}t\cos t+\cos t$$ 1

9. $y''+y'=4\sin^2 t\,,\quad y(0)=y'(0)=1$

$\mathcal{L}[y'']+\mathcal{L}[y']=4\mathcal{L}[\sin^2 t]\,,$ $\qquad\mathcal{L}[y(t)]=Y(s)$

$\mathcal{L}[y'(t)]=sY(s)-1$ 0,5

$\mathcal{L}[y''(t)]=s^2Y(s)-s-1$ 0,5

$$\mathcal{L}[\sin^2 t]=\mathcal{L}\left[\frac{1-\cos 2t}{2}\right]=\frac{1}{2}\mathcal{L}[1]-\frac{1}{2}\mathcal{L}[\cos 2t]$$

$$=\frac{1}{2}\cdot\frac{1}{s}-\frac{1}{2}\frac{s}{s^2+4}$$ 0,5

$$\underline{s^2Y(s)}-s-1+\underline{sY(s)}-1=2\cdot\frac{1}{s}-2\cdot\frac{s}{s^2+4}$$ 0,5

$$Y(s)(s^2+s)=\frac{2}{s}-\frac{2s}{s^2+4}+s+2$$

$$Y(s)=\frac{2}{s(s^2+s)}-\frac{2s}{(s^2+s)(s^2+4)}+\frac{s+2}{s^2+s}$$

$$Y(s)=\frac{2}{s^2(s+1)}-\frac{2}{(s+1)(s^2+4)}+\frac{s+2}{s(s+1)}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = \underbrace{\mathcal{L}^{-1}\left[\frac{2}{s^2(s+1)}\right]}_{I} - \underbrace{2\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2+4)}\right]}_{II} + \underbrace{\mathcal{L}^{-1}\left[\frac{s+2}{s(s+1)}\right]}_{III} \quad 1$$

I: $\frac{1}{s^2(s+1)} = -\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}$

$$-\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = \underline{-1 + t + e^{-t}} \quad 1$$

II: $-2\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2+4)}\right]$

$$\frac{1}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4} = \frac{1}{5}\cdot\frac{1}{s+1} + \frac{1}{5}\,\frac{-s+1}{s^2+4}$$

$$-2\mathcal{L}^{-1}\left[\frac{1}{(s+1)(s^2+4)}\right] = -\frac{2}{5}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \frac{2}{5}\mathcal{L}^{-1}\left[\frac{-s+1}{s^2+4}\right]$$

$$= -\frac{2}{5}\cdot e^{-t} + \frac{2}{5}\mathcal{L}^{-1}\left[\frac{s}{s^2+4}\right] - \frac{2}{5}\mathcal{L}^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= -\frac{2}{5}e^{-t} + \frac{2}{5}\cos 2t - \frac{\cancel{2}}{5}\cdot\frac{1}{\cancel{2}}\sin 2t \quad 1$$

III: $\mathcal{L}^{-1}\left[\frac{s+2}{s(s+1)}\right]$

$$\frac{s+2}{s(s+1)} = \frac{2}{s} - \frac{1}{s+1}$$

$$\Rightarrow \mathcal{L}^{-1}\left[\frac{2}{s}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = 2 - e^{-t} \quad 1$$

$y(t) = I + II + III + \dots$