Površinski integral

I vrste:

- $\iint_S f(x,y,z)\,dS$, S – površ u prostoru

$S: z=f(x,y)$, D je njena projekcija na xOy ravan, tada

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,z(x,y))\sqrt{1+z_x'^2+z_y'^2}\,dx\,dy$$

- Neka je m masa raspoređena po površi S sa gustinom $\rho(x,y,z)$, tada je masa površi S jednaka

$$m(S) = \iint_S \rho(x,y,z)\,dS$$

II vrste: F-ja $f(x,y,z)$ neprekidna na $S: z=z(y,x)$,

gornja strana površi

a) $\iint_S f(x,y,z)\,dx\,dy = \pm \iint_S f(x,y,z(x,y))\,dx\,dy$,

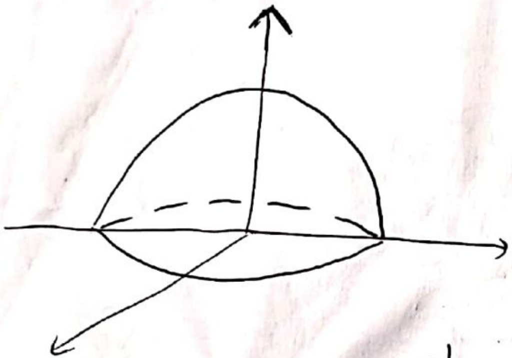
donja strana površi

b) $\iint_S f(x,y,z)\,dx\,dy = \iint_S f(x,y,z)\cos\mu\,dS$

gde je μ ugao koji vektor normale površi S gradi sa z-osom.

1. Izračunati $\iint\limits_S (x^2+y^2)\,dS$ gde je

$S: \quad 0 \le z \le 2-x^2-y^2$



$D: \quad 2-x^2-y^2=0$

$x^2+y^2=2$

$x=\rho\cos\varphi \qquad 0\le\varphi\le 2\pi$

$y=\rho\sin\varphi, \qquad 0\le\rho\le\sqrt{2}$

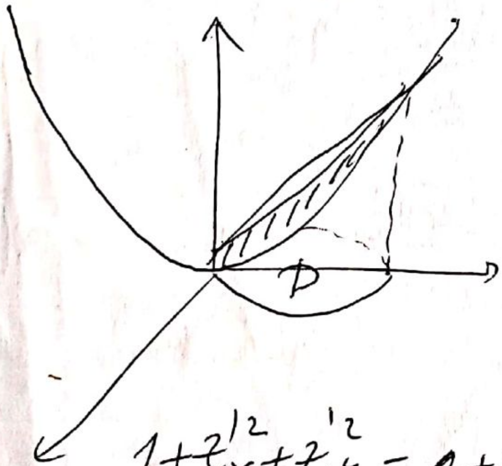
$z'_x=-2x, \quad z'_y=-2y, \qquad |J|=\rho$

$1+z'^2_x+z'^2_y = 1+4(x^2+y^2) = 1+4\rho^2$

$$I=\iint\limits_S (x^2+y^2)\,dS = \int_0^{2\pi} d\varphi \int_0^{\sqrt{2}} \rho^3\sqrt{1+4\rho^2}\,d\rho =$$

$$= 2\pi\,\frac{1}{120}\sqrt{(4x^2+1)^3}\,(6x^2-1)\Big|_0^{\sqrt{2}} = \frac{149}{30}\pi$$

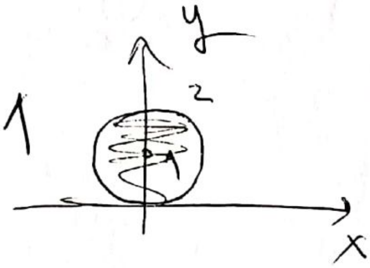
2. $\iint\limits_S \frac{1}{\sqrt{1+4z}}\,dS$, S: konačan deo površi $z=x^2+y^2$ odsečen sa $z=2y$.



$D: \quad x^2+y^2=2y$

$x^2+(y-1)^2=1$

$C(0,1), \; r=1$



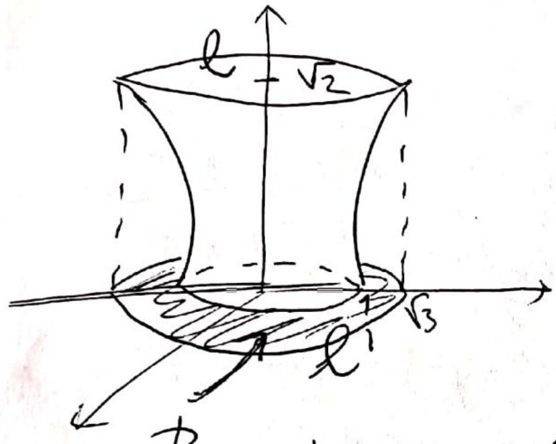
$x=\rho\cos\varphi \qquad 0\le\rho\le 2\sin\varphi$

$y=\rho\sin\varphi \qquad 0\le\varphi\le\pi$

$1+z'^2_x+z'^2_y = 1+4\rho^2$

$$I=\int_0^{\pi} d\varphi \int_0^{2\sin\varphi} \frac{\rho}{\cancel{\sqrt{1+4\rho^2}}}\cancel{\sqrt{1+4\rho^2}}\,d\rho = 2\int_0^{\pi}\sin^2\varphi\,d\varphi = 2\frac{\pi}{2}=\pi$$

3. $\iint\limits_S \frac{1}{\sqrt{2z^2+1}}\,dS$, gde je S deo površi $x^2+y^2-z^2=1$, koji se nalazi između ravni $z=0$ i $z=\sqrt{2}$.



$$l:\begin{cases} x^2+y^2-z^2=1 \\ z=\sqrt{2}\end{cases}$$

$$l:\ x^2+y^2=3,\ z=\sqrt{2}$$

$$l':\ x^2+y^2=3,\ z=0$$

$$D:\ \begin{matrix} x=\rho\cos\varphi \\ y=\rho\sin\varphi,\end{matrix}\quad |J|=\rho,\quad \begin{matrix} 1\le\rho\le\sqrt{3} \\ 0\le\varphi\le 2\pi\end{matrix}$$

$$z_x'=\frac{x}{\sqrt{x^2+y^2-1}},\ z_y'=\frac{y}{\sqrt{x^2+y^2-1}},$$

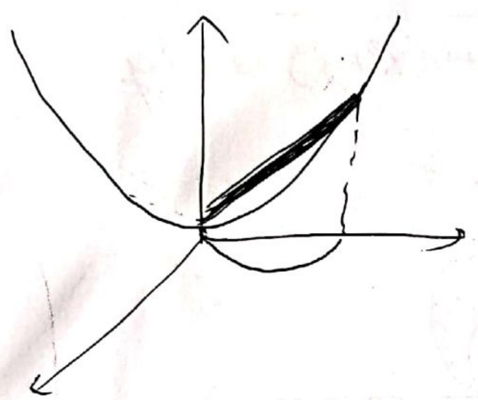
$$1+z_x'^2+z_y'^2=\frac{2(x^2+y^2)-1}{x^2+y^2-1}$$

$$I=\int_0^{2\pi}d\varphi\int_1^{\sqrt{3}}\frac{\rho}{\cancel{\sqrt{2\rho^2-1}}}\cdot\frac{\cancel{\sqrt{2\rho^2-1}}}{\sqrt{\rho^2-1}}\,d\rho=$$

$$=2\pi\int_1^{\sqrt{3}}\frac{\rho}{\sqrt{\rho^2-1}}\,d\rho=2\pi\sqrt{2}$$

(4.) Izračunati masu dela površi $z = x^2 + y^2$ odsečene sa $z = 2y$, ako je gustina data sa

$$\rho(x,y,z) = \frac{y}{\sqrt{1+4z}}.$$



$z = x^2 + y^2, \quad z = 2y$

$x^2 + y^2 = 2y$

$x^2 + (y-1)^2 = 1$

$x = \rho\cos\varphi \qquad 0 \le \varphi \le \pi$

$y = \rho\sin\varphi, \qquad 0 \le \rho \le 2\sin\varphi$

$$m = \iint_S \rho\, dS = \int_0^{\pi} d\varphi \int_0^{2\sin\varphi} \frac{\rho^2 \sin\varphi}{\cancel{\sqrt{1+4\rho^2}}} \cdot \cancel{\sqrt{1+4\rho^2}}\, d\rho$$

$$= \int_0^{\pi} \frac{1}{3} \cdot 8\sin^3\varphi \cdot \sin\varphi\, d\varphi = \frac{8}{3} \int_0^{\pi} \sin^4\varphi\, d\varphi = \frac{8}{3} \cdot \frac{3\pi}{8} = \pi$$

(5.) Izr. masu dela površi $z = x^2 + y^2$ odsečene sa $z = 2x$, ako je $\rho(x,y,z) = \dfrac{x+1}{\sqrt{1+4z}}$

$$m = \iint_S \frac{x+1}{\sqrt{1+4z}}\, dS, \qquad z = x^2 + y^2$$

$1 + z_x'^2 + z_y'^2 = 1 + 4(x^2+y^2)$

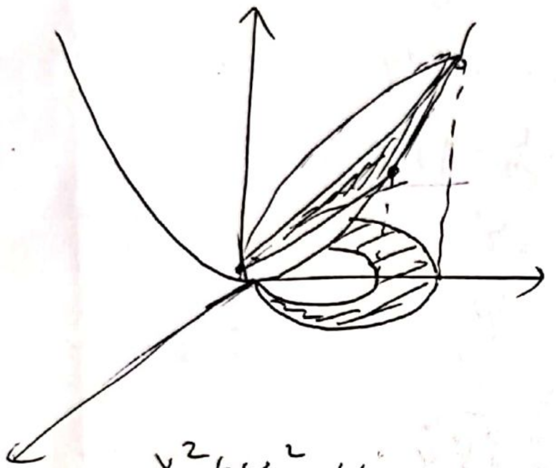
$$= \iint_D \frac{x+1}{\cancel{\sqrt{1+4(x^2+y^2)}}} \cancel{\sqrt{1+4(x^2+y^2)}}\, dx\, dy \qquad dS = \sqrt{1+4(x^2+y^2)}\, dx\, dy$$

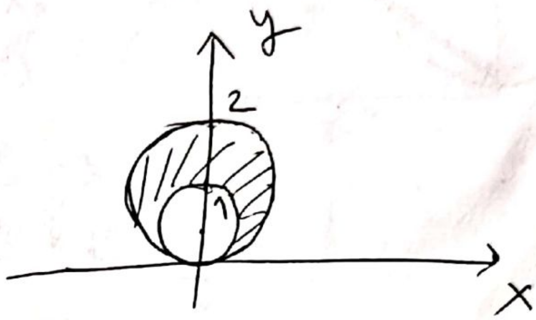
$$= \iint_D (x+1)\, dx\, dy = \int_{-\frac{\pi}{2}}^{\pi/2} d\varphi \int_0^{2\cos\varphi} \rho(\rho\cos\varphi + 1)\, d\rho = \ldots = 2\pi$$

60) Odrediti masu dela površi $z = x^2+y^2$ koja se nalazi između ravni $z=y$ i $z=2y$ ako je gustina $\rho(x,y,z) = \sqrt{1+4z}$

$m = \iint_S \sqrt{1+4z}\,dS$, $\quad 1+z_x'^2+z_y'^2 = 1+4(x^2+y^2)$

$dS = \sqrt{1+4(x^2+y^2)}\,dx\,dy$





$x^2+y^2=y$

$x^2-(y-\frac{1}{2})^2=\frac{1}{4}$

$C(0,\frac{1}{2}),\ r=\frac{1}{2}$

$x^2+y^2=2y$

$x^2+(y-1)^2=1$

$C(0,1),\ r=1$

$x=\rho\cos\varphi$

$y=\rho\sin\varphi$

$\sin\varphi \le \rho \le 2\sin\varphi$

$0 \le \varphi \le \pi$

$$m = \int_0^{\pi} d\varphi \int_{\sin\varphi}^{2\sin\varphi} \sqrt{1+4\rho^2}\cdot\sqrt{1+4\rho^2}\,\rho\,d\rho =$$

$$= \int_0^{\pi} d\varphi \int_{\sin\varphi}^{2\sin\varphi} (1+4\rho^2)\rho\,d\rho = \int_0^{\pi}\left(\frac{3}{2}\sin^2\varphi + 15\sin^4\varphi\right)d\varphi$$

$$= \frac{3}{2}\,\frac{\pi}{2} + 15\,\frac{3\pi}{8} = \frac{51\pi}{8}$$

Scanned with CamScanner